

Infinitely many local higher symmetries without recursion operator or master symmetry: integrability of the Foursov–Burgers system revisited

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January 19, 2008

We consider the Burgers-type system studied by Foursov,

$$\begin{aligned} w_t &= w_{xx} + 8ww_x + (2 - 4\alpha)zz_x, \\ z_t &= (1 - 2\alpha)z_{xx} - 4\alpha zw_x + (4 - 8\alpha)wz_x - (4 + 8\alpha)w^2z + (-2 + 4\alpha)z^3, \end{aligned}$$

for which no recursion operator or master symmetry was known so far, and prove that this system admits infinitely many local higher symmetries that are constructed using a nonlocal *two-term* recursion relation rather than a recursion operator.

Keywords: higher symmetries; recursion relation; recursion operator; master symmetry; C -integrability; linearization

MSC: 37K05; 37K10; 35A30; 58J70; 58J72

Introduction

Following Foursov [9] consider a Burgers-type system

$$\begin{aligned} w_t &= w_{xx} + 8ww_x + (2 - 4\alpha)zz_x, \\ z_t &= (1 - 2\alpha)z_{xx} - 4\alpha zw_x + (4 - 8\alpha)wz_x - (4 + 8\alpha)w^2z + (-2 + 4\alpha)z^3, \end{aligned} \tag{1}$$

where α is a real parameter.

For $\alpha = 0$ system (1) is equivalent [9] to a system found by Svinolupov [33] while for $\alpha = 1$ this system is equivalent to system (4.13) in Olver and Sokolov [26]. Finally, for $\alpha = 1/2$ a recursion operator for (1) was found in [9].

Moreover, for any value of α the differential substitution

$$w = \frac{u_x}{4u}, \quad z = -\frac{v}{2\sqrt{u}}, \tag{2}$$

the inverse of which was found in [34], maps the triangular system

$$u_t = u_{xx} + (1 - 2\alpha)v^2, \quad v_t = (1 - 2\alpha)v_{xx} \tag{3}$$

into (1).

System (3) is triangular and hence C -integrable. C -integrability means here that solving (3) boils down to solving a *linear* heat equation for v , $v_t = (1 - 2\alpha)v_{xx}$, and then solving a *linear* inhomogeneous PDE, namely $u_t = u_{xx} + (1 - 2\alpha)v^2$ for a given v . More generally, C -integrability of a system of PDEs is tantamount to existence of a transformation that reduces this system to a linear (or at least triangular) one, see [4] for further details.

In view of the above system (1) is C -integrable as well, and the general solution of (1) is obtained by plugging the general solution of (3) into (2). However, we are left with an open problem of whether the system (1) has infinitely many local higher symmetries for arbitrary value of α . Recall that geometrically a higher (or generalized [25]) symmetry for a system of PDEs is essentially a solution of linearized version of this system; the solution in question is a vector function on the associated diffiety and, in general, depends on higher (i.e., of order greater than one) jets, see e.g. [3], [11], [15], [17], [25] and references therein for further details. As for generalizations of this concept, see [3], [11], [13], [14], [16], [15], [17] and references therein for nonlocal higher symmetries, and [7], [19], [20], [21], [36] and references therein for higher conditional symmetries.

In a somewhat more technical language, a (local) higher symmetry for (1) can be identified (see e.g. [3], [5], [11], [15], [23], [25]) with a two-component vector $\mathbf{K} = (K^1, K^2)^T$ whose entries K^i depend on x, t, w, z and on a finite number of x -derivatives of w and z up to an order k (which is in general different for different symmetries); here and below the superscript T indicates the transposed matrix. It is further required that the evolution system

$$w_\tau = K^1, \quad z_\tau = K^2 \quad (4)$$

is a commuting flow for (1), i.e.,

$$\frac{\partial^2 w}{\partial \tau \partial t} = \frac{\partial^2 w}{\partial t \partial \tau}, \quad \frac{\partial^2 z}{\partial \tau \partial t} = \frac{\partial^2 z}{\partial t \partial \tau},$$

where the partial derivatives with respect to t and τ are computed using (1) and (4).

For $\alpha \neq 0, 1/2, 1$ system (1) was known to have [9] six higher symmetries but no recursion operator or master symmetry was found so far, so it was not clear whether (1) for $\alpha \neq 0, 1/2, 1$ has infinitely many (rather than just six) higher symmetries. Recall that a recursion operator is, in essence, an operator that maps symmetries into (new) symmetries, see e.g. [2], [3], [5], [11], [17], [25] and references therein. In turn, a master symmetry is essentially a higher symmetry, typically a *nonlocal* one, such that the commutator of this symmetry with any given symmetry yields a (new) symmetry, see e.g. [2], [5], [6], [3], [24], [25] and references therein for details.

In view of the recent results of Sanders and van der Kamp [12] who found several examples of two-component triangular evolution systems that possess only a finite number (greater than one) of local higher symmetries, it is natural to ask whether (1) could provide an example of a *non-triangular* system with finitely many local higher symmetries.

In the present paper we show that this is not the case: system (1) has infinitely many commuting local higher symmetries. Quite unusual, however, is the fact that these symmetries are generated using a *nonlocal two-term recursion relation* (9) rather than a recursion operator or a master symmetry, see Theorem 1 below for details. Moreover, we believe that it is impossible to construct a recursion operator of a reasonably “standard” form that would reproduce at least a part of the hierarchy from Theorem 1, but no proof of this claim is available so far.

Generation of symmetries via recursion relations turns out to be of considerable interest on its own right. To the best of our knowledge, the first examples of this kind have appeared in [1], with the simplest case given by Eqs.(7), (8) below. However, the recursion relation (7) is *local*, and therefore obviously produces local symmetries; for other recursion relations in [1] locality of symmetries is also pretty much immediate, unlike the recursion relation (9) in Theorem 1 below.

In general, if we deal with a recursion relation that involves nonlocalities, then establishing locality and commutativity of the symmetries generated using this relation is a highly nontrivial task. It is important to stress that the hitherto known methods for proving locality of hierarchies of symmetries are based upon existence of a hereditary recursion operator or of a master symmetry, see e.g. [28], [31], [32], [35] and references therein, in general are not applicable in this situation.

For the particular case of recursion relation (9) we proved locality and commutativity of the symmetries \mathbf{K}_n (9) using some *ad hoc* arguments in spirit of [11] and [18], but it would be very interesting to find general, more powerful methods that would not require the existence of scaling symmetry.

It would be of interest to understand geometrical meaning of commutativity of symmetries in this setting. For the “standard” hierarchies this means vanishing of the Nijenhuis torsion of the recursion operator (see e.g. [5]) but it is not quite clear how one could generalize this to the case of the recursion relations.

Finally, perhaps the most important open problem here is whether there exist any S -integrable (i.e., roughly speaking, integrable via the inverse scattering transform, see [4] for details) systems of PDEs that have no “standard” recursion operator or master symmetry but nevertheless possess infinitely many symmetries generated through a recursion relation involving two or more terms.

1 Preliminaries

We start with recalling some basic properties of (3).

First of all, for $\alpha = 1/2$ this system decouples and takes the form

$$u_t = u_{xx}, \quad v_t = 0, \quad (5)$$

i.e., in this case we have a linear homogeneous heat equation for u , and v is simply an arbitrary function of x .

On the other hand, for $\alpha \neq 1/2$ upon introducing a new independent variable $\tau = (1 - 2\alpha)t$ instead of t in (3) and setting $a = 1/(1 - 2\alpha)$ we obtain the system that was studied, *inter alia*, in [1], viz.

$$u_\tau = au_{xx} + v^2, \quad v_\tau = v_{xx}. \quad (6)$$

Moreover, system (6) (and hence (3)) has infinitely many symmetries. Indeed, by Theorem 2.2 of [1] the system in question has infinitely many local higher symmetries of the form

$$\mathbf{G}_n = \begin{pmatrix} b_n u_n + Q_n \\ v_n \end{pmatrix}, \quad (7)$$

where $b_n = b_{n-1} - (1 - a)b_{n-2}/2$ with $b_1 = 1$, $b_2 = a$, and

$$Q_n = D_x Q_{n-1} - \frac{1-a}{2} D_x^2 Q_{n-2} + v v_{n-2}. \quad (8)$$

The initial conditions for the recursion (8) are $Q_1 = 0$, $Q_2 = v^2$.

Here u_i and v_i stand for the i^{th} x -derivatives of u and v , $u_0 \equiv u$, $v_0 \equiv v$, and D_x denotes the total x -derivative (see e.g. [3], [11], [23], [25])

$$D_x = \frac{\partial}{\partial x} + \sum_{i=0}^{\infty} \left(u_{i+1} \frac{\partial}{\partial u_i} + v_{i+1} \frac{\partial}{\partial v_i} \right).$$

Recall that a function f of $x, t, u_0, v_0, u_1, v_1, \dots$ is said to be *local* if it depends only on x, t , and a *finite* number of u_i and v_i , see e.g. [3], [23], [25] and references therein for further details.

Note that for $a = 1$ (i.e., $\alpha = 0$) a recursion operator for (6) was found by Oevel in [24], so for $\alpha = 0$ we can readily find a recursion operator for (1) from the Oevel's recursion operator. On the other hand, for $\alpha = 1/2$ (i.e., $a = 0$) Foursov [9] also found a recursion operator for (1).

However, for generic a no recursion operator is known for (6). Moreover, for generic a the form of leading coefficients of higher-order symmetries of (6), see (7), appears to preclude existence of any “reasonable” recursion operator for (6), and thus for (3) and (1) as well, but so far we were unable to prove this claim. The supposed nonexistence of recursion operator should be closely related to the number-theoretic aspects of the symmetry analysis for (6), see [1], [29] and references therein.

2 Infinitely many local higher symmetries for (1)

Using the inverse of the differential substitution (2) (of course, the inverse in question should be considered as a covering in the sense of [3], [14], [15], [16], [17] and references therein) we can obtain infinitely many symmetries for (1) with an arbitrary value of α different from $1/2$ from the symmetries of (6) given by (7), (8). Most importantly, all of these symmetries are local by virtue of the following result.

Theorem 1 *For $\alpha \neq 1/2$ system (1) has infinitely many commuting local higher symmetries \mathbf{K}_n generated using the nonlocal two-term recursion relation*

$$\mathbf{K}_n = \begin{pmatrix} D_x + 4w + 4w_x D_x^{-1} & 0 \\ 2(z_x - 2wz) D_x^{-1} & D_x + 2w \end{pmatrix} \mathbf{K}_{n-1} + M \mathbf{K}_{n-2} \quad (9)$$

starting from

$$\mathbf{K}_1 = \begin{pmatrix} w_x \\ z_x \end{pmatrix}, \quad \mathbf{K}_2 = \begin{pmatrix} -\frac{w_{xx} + 8ww_x}{2\alpha - 1} + 2zz_x \\ z_{xx} + 4wz_x - \frac{4z(\alpha w_x + (2\alpha + 1)w^2)}{2\alpha - 1} - 2z^3 \end{pmatrix}. \quad (10)$$

Here

$$M = \begin{pmatrix} M_{11} & zD_x + z_x \\ M_{21} & -2z^2 \end{pmatrix},$$

$$M_{11} = -\frac{1}{2\alpha - 1} (\alpha D_x^2 + 8\alpha w D_x + 2(\alpha(6w_x + 8w^2) - (2\alpha - 1)z^2) + 4(\alpha(w_{xx} + 8ww_x) - (2\alpha - 1)zz_x) D_x^{-1}),$$

$$M_{21} = \frac{2\alpha}{2\alpha - 1} z D_x + \frac{16\alpha}{2\alpha - 1} w z + 4 \left(\frac{2\alpha}{2\alpha - 1} (w_x + 4w^2) z - z^3 \right) D_x^{-1}.$$

We defer the proof of this theorem until the next section.

Recall (see Introduction) that locality of the symmetries \mathbf{K}_j , $j \in \mathbb{N}$, means that they depend only on x, t, w, z and a *finite* number of their x -derivatives w_i and z_i (w_i and z_i stand for the i^{th} x -derivatives of w and z , and we set $w_0 \equiv w$ and $z_0 \equiv z$ for convenience) and do not involve any nonlocal quantities, cf. e.g. [3], [15], [17], [23], [25].

The total x -derivative D_x now takes the form

$$D_x = \frac{\partial}{\partial x} + \sum_{i=0}^{\infty} \left(w_{i+1} \frac{\partial}{\partial w_i} + z_{i+1} \frac{\partial}{\partial z_i} \right).$$

Let $f \in \text{Im} D_x$ be a polynomial in a finite number of w_i and z_i with zero free term (let us stress that f is not allowed to depend explicitly on x and t). In Theorem 1 and below we make a *blanket assumption* that the result of action of D_x^{-1} on any such f again is a polynomial in a finite number of w_i and z_i with zero free term, i.e., we always set the integration constant to zero. Note that an alternative (and more general) approach is to define the operator D_x^{-1} in the fashion described in [10], [22], [30].

With the above assumption in mind we readily find that the first nontrivial symmetry generated via (9), namely, \mathbf{K}_3 , has the form $\mathbf{K}_3 = (K_3^1, K_3^2)^T$, where

$$\begin{aligned} K_3^1 &= -\frac{(\alpha+1)}{2\alpha-1} w_{xxx} - \frac{12(\alpha+1)}{2\alpha-1} w w_{xx} + 3z z_{xx} - \frac{12(\alpha+1)}{2\alpha-1} w_x^2 \\ &\quad + 3z_x^2 - 6z^2 w_x - \frac{48(\alpha+1)}{2\alpha-1} w^2 w_x + 12w z z_x, \\ K_3^2 &= z_{xxx} + \frac{6\alpha}{2\alpha-1} z w_{xx} + 6w z_{xx} + 6w_x z_x + \frac{12(4\alpha+1)}{2\alpha-1} w z w_x + 6(2w^2 - z^2) z_x \\ &\quad + \frac{24(2\alpha+1)}{2\alpha-1} w^3 z - 12w z^3. \end{aligned}$$

We see that the symmetries \mathbf{K}_j , $j = 1, 2, 3$ are independent of x and t . Moreover, it is easily seen that by virtue of the above definition of action of D_x^{-1} the symmetries \mathbf{K}_j in fact do not depend explicitly on x and t for all $j \in \mathbb{N}$.

3 Proof of Theorem 1

The recursion relation (8) can be rewritten in the terms of symmetries \mathbf{G}_i from (7) as

$$\mathbf{G}_n = D_x(\mathbf{G}_{n-1}) - \begin{pmatrix} (1-a)/2 & 0 \\ 0 & 0 \end{pmatrix} D_x^2(\mathbf{G}_{n-2}) + \begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix} \mathbf{G}_{n-2}, \quad (11)$$

and the recursion relation (9) readily follows from (11) and (2).

Now we have to show that the symmetries \mathbf{K}_j generated via the recursion relation (9) with the initial data (10) are *local* in the sense of the preceding section for all $j = 3, 4, \dots$. This obviously boils down to proving that they do not involve nonlocalities like $y = D_x^{-1}(w)$.

To this end we shall use induction on n starting from $n = 1$. In view of the specific form of the recursion relation (9) we only have to show that once $\mathbf{K}_{j-1} = (K_{j-1}^1, K_{j-1}^2)^T$ and $\mathbf{K}_{j-2} = (K_{j-2}^1, K_{j-2}^2)^T$

are such that $K_{j-1}^1, K_{j-2}^1 \in \text{Im}D_x$ (and hence \mathbf{K}_j is local) then we have $K_j^1 \in \text{Im}D_x$. This is obviously true for $j = 3$, so we only need to establish validity of the induction step. This is done using the following result.

Lemma 1 *Fix $j \geq 3$ and assume that \mathbf{K}_i , $i = j-2, j-1$, are local, and $K_{j-1}^1, K_{j-2}^1 \in \text{Im}D_x$. Then \mathbf{K}_j defined via (9) is local too, and we have $K_j^1 \in \text{Im}D_x$.*

Proof of the lemma. As $K_{j-1}^1, K_{j-2}^1 \in \text{Im}D_x$ by assumption, locality of \mathbf{K}_j is immediately inferred from (9), so it remains to prove that $K_j^1 \in \text{Im}D_x$.

Now, the condition $K_j^1 \in \text{Im}D_x$ can be restated as follows: $\rho_j \in \text{Im}D_x$, where $\rho_j \equiv \rho'_0[\mathbf{K}_j]$, $\rho_0 = w$ is a conserved density for (1), and $f'[\mathbf{K}]$ stands (cf. e.g. [2], [5], [24]) for the directional derivative (also known as linearization, see e.g. [3], [15], [17]) of f along $\mathbf{K} = (K^1, K^2)^T$:

$$f'[\mathbf{K}] = \sum_{i=0}^{\infty} \left(\frac{\partial f}{\partial w_i} D_x^i(K^1) + \frac{\partial f}{\partial z_i} D_x^i(K^2) \right).$$

Recall (see e.g. [3], [11], [23], [25]) that a local function ρ is said to be a local *conserved density* for (1) if $D_t(\rho) \in \text{Im}D_x$. A local conserved density ρ is said to be *nontrivial* if $\rho \notin \text{Im}D_x$, and *trivial* otherwise.

Next, as \mathbf{K}_j is a symmetry of (1) by construction, the quantity ρ_j is a local conserved density for (1), see e.g. [2], [11], [3], [25].

However, for $\alpha \neq 1, 1/2$ the quantity ρ_0 is the only nontrivial local conserved density for (1) modulo the terms from $\text{Im}D_x$. Indeed, by virtue of the results of [23] (see also Theorem 5-1 of [8], and cf. [27] and references therein) any nontrivial local conserved density for (1) depends (again modulo the terms from $\text{Im}D_x$) only on $x, t, w, z, w_x, z_x, w_{xx}, z_{xx}$, and the direct search for all conserved densities of this form proves our claim.

Hence the most general local conserved density for (1) has the form $c\rho_0 + \tilde{\rho}$, where $\tilde{\rho} \in \text{Im}D_x$ and c is a constant. In particular, if \mathbf{K}_j is local then we have

$$\rho_j = c_j \rho_0 + \tilde{\rho}_j, \tag{12}$$

where $\tilde{\rho}_j \in \text{Im}D_x$ and c_j are constants.

In analogy with Krasil'shchik [18], let us show that if \mathbf{K}_j is local then $c_j = 0$, and thus $\rho_j \in \text{Im}D_x$.

First of all, note that system (1) has [9] a scaling symmetry¹

$$\mathbf{S} = 2t(1 - 2\alpha)\mathbf{K}_2 + x\mathbf{K}_1 + \begin{pmatrix} w \\ z \end{pmatrix},$$

and it is readily seen that all \mathbf{K}_j constructed using (9) are \mathbf{S} -homogeneous: we have $L_{\mathbf{S}}(\mathbf{K}_j) \equiv [\mathbf{S}, \mathbf{K}_j] = j\mathbf{K}_j$, $j = 1, 2, \dots$. Here $L_{\mathbf{Q}}$ denotes the Lie derivative along \mathbf{Q} , see e.g. [2], [25], [28], [32] for details, and $[\cdot, \cdot]$ is the standard commutator of symmetries (see e.g. [2], [3], [5], [11], [17], [23], [25]):

$$[\mathbf{F}, \mathbf{G}] = \mathbf{G}'[\mathbf{F}] - \mathbf{F}'[\mathbf{G}]. \tag{13}$$

¹Note that \mathbf{S} , \mathbf{K}_1 , and \mathbf{K}_2 exhaust all linearly independent Lie point (or, more precisely, higher symmetries that are equivalent to the Lie point ones in the sense of [11, 25]) symmetries of (1).

Hence $L_{\mathbf{S}}(\rho_j) = \rho'_j[\mathbf{S}] = (j+1)\rho_j + \eta_j$, where $\eta_j \in \text{Im}D_x$. On the other hand,

$$L_{\mathbf{S}}(c_j\rho_0 + \tilde{\rho}_j) = c_j\rho_0 + \zeta_j,$$

where $\zeta_j \in \text{Im}D_x$. Therefore, if we act by $L_{\mathbf{S}}$ on the left- and right-hand side of (12) and equate the resulting expressions, we obtain

$$jc_j\rho_0 + \theta_j = 0, \tag{14}$$

where $\theta_j \in \text{Im}D_x$.

As $\rho_0 \notin \text{Im}D_x$ and $j \neq 0$ by assumption, Eq.(14) can hold only if $c_j \equiv 0$. Hence $\rho_j = \tilde{\rho}_j \in \text{Im}D_x$, and the lemma is proved. \square

Now that we have proved locality of \mathbf{K}_j for all $j \in \mathbb{N}$, we can easily prove commutativity of \mathbf{K}_j with respect to the bracket (13): $[\mathbf{K}_i, \mathbf{K}_j] = 0$, $i, j = 1, 2, 3, \dots$

First of all, in analogy with the reasoning presented in Chapter 4 of [11] for scalar evolution equations (cf. also [3], [18], [15]) it can be shown that any \mathbf{S} -homogeneous x, t -independent local higher symmetry \mathbf{G} of (1) of order $j \geq 1$ is of the form

$$\mathbf{G} = \begin{pmatrix} \alpha_j w_j + \tilde{G}_j^1 \\ \beta_j z_j + \tilde{G}_j^2 \end{pmatrix}, \tag{15}$$

where α_j, β_j are constants, and $\tilde{G}_j^1, \tilde{G}_j^2$ are polynomials in $w, z, w_1, z_1, \dots, w_{j-1}, z_{j-1}$, and these polynomials have no free terms and no linear terms.

Let \mathcal{L}_k be the space of \mathbf{S} -homogeneous x, t -independent local higher symmetries of (1) of order no greater than k . By the above, there exists a basis in \mathcal{L}_k that consists of symmetries of the form (15) for $j = 1, 2, \dots, k$ (note that in principle this basis may contain more than one symmetry of given order j).

Clearly, $\mathbf{K}_i \in \mathcal{L}_i$ and $[\mathbf{K}_i, \mathbf{K}_j] \in \mathcal{L}_{i+j}$. But using (13) and (15) we readily see that the commutator $[\mathbf{K}_i, \mathbf{K}_j]$ contains no linear terms, and hence (cf. e.g. [11]) this commutator can belong to \mathcal{L}_{i+j} only if $[\mathbf{K}_i, \mathbf{K}_j] = 0$. Thus, $[\mathbf{K}_i, \mathbf{K}_j] = 0$ for all $i, j \in \mathbb{N}$, and this completes the proof of Theorem 1.

Acknowledgements

This research was supported in part by the Czech Grant Agency (GAČR) under grant No. 201/04/0538, by the Ministry of Education, Youth and Sports of the Czech Republic (MŠMT ČR) under grant MSM 4781305904, and by Silesian University in Opava under grant IGS 9/2008. It is my great pleasure to thank B. Kruglikov, M. Marvan and T. Tsuchida for useful suggestions.

References

- [1] F. Beukers, J.A. Sanders, J.P. Wang, On Integrability of Systems of Evolution Equations, *J. Diff. Eq.* **172** (2001), 396–408.
- [2] M. Błaszak, *Multi-Hamiltonian Theory of Dynamical Systems*, Springer, Heidelberg, 1998.

- [3] A.V. Bocharov et al., *Symmetries and conservation laws for differential equations of mathematical physics*. Edited and with a preface by I.S. Krasil'shchik and A.M. Vinogradov, American Mathematical Society, Providence, RI, 1999.
- [4] F. Calogero, Why are certain nonlinear PDEs both widely applicable and integrable?, in: *What is Integrability?*, ed. V.E. Zakharov, Springer, New York, 1991, pp. 1–62.
- [5] I. Dorfman, *Dirac Structures and Integrability of Nonlinear Evolution Equations*, John Wiley & Sons, Chichester, 1993.
- [6] A.S. Fokas and B. Fuchssteiner, The hierarchy of the Benjamin–Ono equation, *Phys. Lett. A* **86** (1981), 341–345.
- [7] A.S. Fokas, Q.M. Liu, Generalized conditional symmetries and exact solutions of non-integrable equations. *Theor. Math. Phys.* **99** (1994), no. 2, 571–582.
- [8] K. Foltinek, Conservation Laws of Evolution Equations: Generic Nonexistence, *J. Math. Anal. Appl.* **235** (1999), 356–379.
- [9] M.V. Foursov, On integrable coupled Burgers-type equations, *Physics Letters A* **272** (2000) 57–64.
- [10] G.A. Guthrie, Recursion operators and non-local symmetries, *Proc. Roy. Soc. London Ser. A* **446** (1994), No.1926, 107–114.
- [11] N.H. Ibragimov, *Transformation Groups Applied to Mathematical Physics*, Reidel, Dordrecht 1985.
- [12] P.H. van der Kamp, J.A. Sanders, Almost integrable evolution equations, *Selecta Math. (N.S.)* **8** (2002), no. 4, 705–719.
- [13] B.G. Konopelchenko, V.G. Mokhnachev, On the group-theoretical analysis of differential equations. *Soviet J. Nuclear Phys.* **30** (1979), no. 2, 559–567.
- [14] I.S. Krasil'shchik, A.M. Vinogradov, Nonlocal symmetries and the theory of coverings: an addendum to A. M. Vinogradov's Local symmetries and conservation laws, *Acta Appl. Math.* **3** (1984), 79–96.
- [15] I.S. Krasil'shchik, V.V. Lychagin, A.M. Vinogradov, *Geometry of jet spaces and nonlinear partial differential equations*, Gordon and Breach, New York, 1986.
- [16] I.S. Krasil'shchik, A.M. Vinogradov, Nonlocal trends in the geometry of differential equations: symmetries, conservation laws, and Bäcklund transformations. Symmetries of partial differential equations, Part I. *Acta Appl. Math.* **15** (1989), no. 1-2, 161–209.
- [17] I.S. Krasil'shchik, P.H.M. Kersten, *Symmetries and recursion operators for classical and supersymmetric differential equations*. Kluwer Academic Publishers, Dordrecht, 2000.

- [18] I. Krasil'shchik, A simple method to prove locality of symmetry hierarchies, Preprint DIPS 9/2002, available online at <http://www.diffiety.org/>
- [19] B. Kruglikov, V. Lychagin, Mayer brackets and solvability of PDEs II, *Trans. A.M.S.* **358** (2006), no. 3, 1077–1103.
- [20] B. Kruglikov, V. Lychagin, Compatibility, multi-brackets and integrability of systems of PDEs, arXiv: math/0610930
- [21] B. Kruglikov, Symmetry approaches for reductions of PDEs, differential constraints and Lagrange-Charpit method, *Acta Appl. Math.* **101** (2008), 145–161 (arXiv: 0712.3425).
- [22] M. Marvan, Another look on recursion operators, in: *Differential Geometry and Applications (Brno, 1995)*, eds. J. Janyška et al., Masaryk University, Brno, 1996, pp. 393–402, available online at <http://www.emis.de/proceedings>
- [23] A.V. Mikhailov, A.B. Shabat and V.V. Sokolov, The symmetry approach to classification of integrable equations, in: *What is Integrability?*, ed. V.E. Zakharov, Springer, New York, 1991, pp. 115–184.
- [24] W. Oevel, *Rekursionmechanismen für Symmetrien und Erhaltungssätze in Integrierten Systemen*, Ph.D. thesis, University of Paderborn, Paderborn, 1984.
- [25] P.J. Olver, *Applications of Lie Groups to Differential Equations*, Springer, New York, 1993.
- [26] P.J. Olver, V.V. Sokolov, Integrable evolution equations on associative algebras, *Commun. Math. Phys.* **193** (1998) 245–268.
- [27] R.O. Popovych and A.M. Samoilenko, Local conservation laws of second-order evolution equations, *J. Phys. A: Math. Theor.* **41** (2008), 362002, 11 p. (arXiv:0806.2765).
- [28] J.A. Sanders and J.P. Wang, Integrable Systems and their Recursion Operators, *Nonlinear Analysis* **47** (2001), no.8, 5213–5240.
- [29] J.A. Sanders and J.P. Wang, Number Theory and the Symmetry Classification of Integrable Systems, in: *Integrability*, ed. A.V. Mikhailov, Springer, Berlin etc., 2009, p.89–118; draft version available online at http://www.cs.vu.nl/~jansa/ftp/WORK100/Chapter2_SandersWang.pdf
- [30] A. Sergyeyev, On recursion operators and nonlocal symmetries of evolution equations, in: *Proc. Sem. Diff. Geom.*, ed. D. Krupka, Silesian University in Opava, Opava, 2000, pp. 159–173 (arXiv: nlin.SI/0012011).
- [31] A. Sergyeyev, On sufficient conditions of locality for hierarchies of symmetries of evolution systems, *Rep. Math. Phys.* **50** (2002), no.3, 307–314.
- [32] A. Sergyeyev, Why nonlocal recursion operators produce local symmetries: new results and applications, *J. Phys. A: Math. Gen.* **38** (2005), 3397–3407 (arXiv: nlin.SI/0410049).

- [33] S.I. Svinolupov, On the analogues of the Burgers equation, *Phys. Lett. A* **135** (1989) 32–36.
- [34] T. Tsuchida and T. Wolf, Classification of polynomial integrable systems of mixed scalar and vector evolution equations: I, *J. Phys. A: Math. Gen.* **38** (2005), 7691–7733 (arXiv: nlin.SI/0412003).
- [35] J.P. Wang, Lenard scheme for two-dimensional periodic Volterra chain, arXiv:0809.3899
- [36] R.Z. Zhdanov, Conditional Lie-Bäcklund symmetry and reduction of evolution equations, *J. Phys. A* **28** (1995), no. 13, 3841–3850 (arXiv: solv-int/9505006).